Solving Difficult SAT Instances Using Greedy Clique Decomposition *

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Abstract. We are dealing with solving difficult SAT instances in this paper. We propose a method for preprocessing SAT instances (CNF formulas) by using consistency techniques known from constraint programming methodology and by using our own consistency technique based on clique decomposition of a graph representing conflicts in the input formula. If the clique decomposition is of a good quality (cliques are appropriately large) it then allows us to make a strong reasoning over the SAT instance, which can in some cases even decide the satisfiability of the SAT instance without search. We implemented our preprocessing method in C++ and compared it with several state-of-the-art SAT solvers on selected difficult SAT instances. The result was a speedup in the order of magnitude compared to the tested SAT solvers.

Keywords: SAT, search, consistency, clique, difficult instances

1 Introduction

The source of inspiration for this paper was a recent work [28] on artificial intelligence planning problems [3]. We exploit the newly developed techniques proposed in [28] for solving *Boolean satisfiability problems (SAT)*. In [28] the problem of finding supporting actions for a goal in the AI planning context is studied. The problem is called a *supports problem* in short. This is some kind of an important sub-problem which must be solved many times when solving AI planning problems using the *planning graphs* [6]. It was shown that the supports problem is NP-complete. In doing so a conversion of an instance of the SAT problem to the instance of the supports problem was used [28]. This proof uncovered some interesting similarities between the SAT problem and the supports problem. Strictly speaking the similarities itself are neither interesting nor useful. They become more interesting after connecting them with the new method for solving supports problems based on a greedy clique decomposition which was also proposed in the mentioned work. The positive experience made with

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the method on planning problems and the observed similarity lead us to the idea of adapting the technique of the greedy clique decomposition to solve SAT problems.

Boolean formula satisfaction problems and SAT solving techniques play an extremely important role in theoretical computer science as well as in practice. The question of whether there exist a complete polynomial time SAT solver is a key question for theoretical computer science and is open for many years (the *P vs. NP problem*) [7]. On the other hand the practical use of SAT problems and SAT solvers in real life applications is also very intensive. Applications of SAT solving techniques range from microprocessor verification [30] and field-programmable gate array design [23] to solving AI planning problems by translating them into Boolean formulas [17].

An excellent performance breakthrough was done in solving SAT problems over the past years. Thanks to new algorithms and implementation techniques focused on real life SAT problems many of the today's benchmark problems [18, 25] are solved by state-of-the-art solvers [11, 12, 14, 15, 21, 27] in time proportional to the size of the input. It seems that the difficulty of many SAT benchmark problems consists in their size only. A lot of smaller benchmark problems are solved in real-time by today's state-of-the-art SAT solvers. The observation that can be deduced upon these facts is that there is almost no chance to compete with the best SAT solvers by a newly written SAT solver on these problems. That is why we are concentrating on difficult instances of SAT problems only, where the word difficult means difficult for today's state-of-the-art SAT solvers.

A very valuable set of difficult (in the mentioned sense) problems was collected by Aloul [1]. Although these problems are small in the length of the input formula they are difficult to be answered. The detailed discussion about hardness of these problems is given in [2]. One of the aspects of problem difficulty is that these problems are mostly unsatisfiable (and this fact is well hidden in the problem). The solver cannot guess a solution using its advanced techniques and heuristics in such a case and it must really perform some search in order to prove that there is no solution. In the case of a positive answer the satisfying valuation of variables serves a witness (of small size) certifying existence of at least one solution. If the solver obtains (possibly by guessing) a witness its task is finished. In contrast to this, there is no such small witness in the case of a negative answer so the search must be performed.

Our contribution to solving SAT problems consists of preprocessing and reformulating of the input Boolean formula in the *CNF* (*conjunctive normal form* - conjunction of disjunctions). The result of this processing is the answer whether the input formula is unsatisfiable or a new formula (hopefully simpler) with the same set of satisfying valuations as that of the input one. If the input formula is not decided by the preprocessing phase then the preprocessed formula is sent to the SAT solver of the user's choice. The idea behind this process is to make the task for the SAT solver easier by deciding the input formula within the fast preprocessing phase or by providing an equivalent but simpler formula to the SAT solver. Experiments showed that the solving process over the above mentioned difficult SAT benchmarks speeds up by the order of magnitude after using our approach.

The reformulation within the preprocessing phase itself is simple. We are viewing the input Boolean formula in CNF as a graph (with vertices and edges). For each *literal* (variable or its negation) of the input formula we consider a vertex and for each conflict between literals we consider an edge. Conflicting literals are those that cannot

be both satisfied in a single valuation of variables, for example positive and negative literals of the same variable are conflicting. Generally, a set of literals of a formula is conflicting if the formula entails that at most one of the literals can be *true*. To be able to use our reasoning based on the clique decomposition we need a graph with appropriately large *complete sub-graphs* (cliques). That is, we need some kind of a good approximation of the sets of conflicting literals. Unfortunately the graph arising from the above interpretation of the Boolean formula in *CNF* is rather sparse (the largest clique is of size 2). That is why we apply further inference by which we deduce more conflicts between the literals and which allow us to introduce more edges into the graph. We are using singleton arc-consistency [5] as the inference technique for deducing new edges.

Having the graph constructed from the input CNF formula, a clique decomposition of this graph is found by a greedy algorithm (we do not need an optimal clique decomposition; we need just some of the reasonable quality). The important property of the constructed clique decomposition is that at most one literal from each clique can be assigned the value *true*. In this situation we perform some kind of literal contribution counting to rule out the literals that can never be *true*. To do this, the maximum number of satisfied clauses by literals of each clique is calculated. Then a literal of a certain clique can be ruled out if the literals from the other cliques together with the selected literal do not satisfy enough clauses to satisfy the input formula.

Although this problem reformulation seems weak it provides a strong reasoning about the dependencies among clauses of the CNF Boolean formula and about the effect of the selection of a value for a variable on the overall satisfiability of the formula. Moreover if all the literals are ruled out during the preprocessing phase the input formula is obviously unsatisfiable. Experimental evaluation showed that this happen frequently on difficult SAT problems. For other cases a new formula in the CNF equivalent to the input formula is produced. The new formula is constructed from the original one by adding clauses that capture all the dependencies inferred by the initial singleton arc-consistency stage and by the literal contribution counting based on the clique decomposition.

The paper is organized as follows. A detailed formal description of the reformulation of a SAT instance using the greedy clique decomposition is given in section 2. The subsequent section 3 is devoted to some experimental comparison of our approach with the existing state-of-the-art SAT solvers. We are discussing the contribution of our method within this section too. Finally we put our work in relation to similar works in the field of Boolean satisfiability and we propose some future research directions of the studied topic.

2 SAT Reformulation Using Greedy Clique Decomposition

We will formally describe the details of the process of SAT problem reformulation in this section. Let $B = \wedge_{i=1}^{n} \vee_{j=1}^{m_i} x_j^i$ be the input Boolean formula in CNF where x_j^i is a literal (variable or its negation) for all possible i and j. A sub-formula $\vee_{j=1}^{m_i} x_j^i$ of the input formula B for every possible i is called a clause. The ith clause of the formula B will be denoted as C_i in the following paragraphs. As it was mentioned in

the introduction, the basic idea of the SAT problem reformulation consists in viewing the input formula as an undirected graph in which the internal structure of the formula is captured in some way. To be more particular the graph will capture the pairs of conflicting literals and it will be constructed in several stages.

2.1 Inference of Conflicting Literals

We start by the construction of an undirected graph $G_B^1 = (V_B^1, E_B^1)$ which will represent trivially conflicting literals. The graph will be called a *graph of trivial conflicts*. The graph G_B^1 will then undergo some further inference process by which the additional conflicts will be inferred. We will denote the resulting undirected graph as $G_B^2 = (V_B^2, E_B^2)$ and call it an *intermediate graph of conflicts*.

The construction of the undirected graph G_B^1 is simple. A vertex is introduced into the graph G_B^1 for each literal occurring in the formula B, that is $V_B^1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} x_j^i$ (notice that $|V_B^1|$ is typically smaller than the length of the formula, since literals may occur many times in the formula while only once in the graph). The construction of the set of edges E_B^1 is also straightforward. An edge $\{x_j^i, x_l^k\}$ is introduced into the graph G_B^1 if the literals x_j^i are x_l^k are trivially conflicting, that is if one is a variable v and the other is $\neg v$ for some Boolean variable v. The graph G_B^1 is finished by performing the above step for all possible pairs of conflicting literals. The interpretation of the graph of conflicts is that if a literal corresponding to a vertex is selected to be assigned the value *true* all literals corresponding to the neighboring vertices must be assigned the value *false*.

An example graph resulting from the described process over a selected benchmark problem is shown in the left part of figure 1. The resulting graph is visibly sparse, since there are edges only between the literals of the same variable. Hence it is not a good starting point for our method and a further inference mechanism for discovering more conflicting pairs of literals (more edges for the graph) must be applied. This further inference mechanism takes the already constructed graph G_B^1 and augments it by adding new edges. The result of this stage is an intermediate graph of conflicts G_B^2.

The process of construction of graph G_B^2 exploits techniques known from standard SAT resolution approaches and from *constraint programming* [9] - *unit propagation* [10, 31], *arc-consistency* (*AC*) [20] and *singleton arc-consistency* (*SAC*) [5]. Before describing the construction of the graph G_B^2 let us recall a modification of notions. We modify the above concepts slightly for the SAT domain to prepare them for our purposes. The following definitions assume the input formula B in CNF and a corresponding graph of conflicts G_B (for example the graph G_B^1 expressing the trivial conflicts).

Definition 1 (Arc-consistency in SAT instance w.r.t. the graph of conflicts). Consider two clauses C_i and C_k for $i,k \in \{1,2,\ldots,n\}$, $i \neq k$ of the formula B. A literal x_j^i ($j \in \{1,2,\ldots,m_i\}$) from the clause C_i is *supported* by the clause C_k with respect to the given graph of conflicts G_B if there exists a literal x_l^k ($l \in \{1,2,\ldots,m_k\}$) from the clause C_k, such that the literals x_j^i and x_l^k are not in a conflict with respect to the graph G_B (not connected by an edge). An ordered pair of clauses (C_i, C_k) of the formula B is called an *arc* in this context. An arc (C_i, C_k) for some $i,k \in \{1,2,\ldots,n\}$

is *consistent* (*or arc-consistent*) with respect to the graph of conflicts G_B if all the literals of the clause C_i are supported by the clause C_k with respect to the graph of conflicts G_B. The formula B is called *arc-consistent* with respect to the graph of conflicts G_B if all the arcs (C_i, C_k) for all $i,k = 1,2,\ldots,n$ are arc-consistent with respect to the graph of conflicts G_B. □

Let us note that our definition is based on a dual view of the satisfaction problem. That is, we use the clauses of the formula as the CSP variables [9] instead of the original Boolean variables. Having these CSP variables, (CSP) constraints necessary for the definition of arc-consistency arise naturally.

The reason for the definition of arc-consistency is that the literals which are not supported according to the definition cannot be assigned the value *true* (this means that the corresponding variable cannot be assigned the value *false* in the case of a negative literal). So the solver can rule out such literals from further attempts to assign them the value *true*, which may reduce the size of the search space. Notice that the definition has the graph of conflicts G_B as a parameter. It is possible to put any correct graph of conflicts as a parameter of this definition, whereas correct means, that if $\{y,z\}$ is the edge of the graph then $B \Rightarrow y \neq z$ must be a tautology. This is obviously true for the graph of trivial conflicts G_B^1. Notice also that if we use the graph of trivial conflicts G_B^1 the definition becomes identical to unit propagation [10, 31].

Having the Boolean formula B the question is how to make it arc-consistent with respect to the given graph of conflicts. For this purpose we adopt techniques developed in constraint programming and by SAT community, namely the arc-consistency enforcing algorithms [9, 20] and unit propagation [10, 31]. There is a great variety of such algorithms, however their common feature is the search for supports for every value (literal) which is suspected of not being supported. The main difference among these algorithms is the efficiency of the search for supports. If an unsupported literal is detected it is ruled out. Ruling out an unsupported literal may cause that some other literal loses its only support. This chain-like propagation of changes continues until a stable state is reached. For purposes of the SAT domain this propagation process is usually augmented by an additional simplification rule. If the consistency enforcing algorithm detects that within some clause there is only one literal that can be selected to be *true*, it is fixed to value *true* and the corresponding clause is cut out from further reasoning (this is exactly the simplification rule from unit propagation).

Unfortunately the defined arc-consistency over Boolean formulas in the CNF form is too weak to infer significantly more conflicts than that are already present in the graph of trivial conflicts. Therefore we need to make the consistency stronger. Perhaps the simplest way to do this is to make the selected consistency technique *singleton* [5]. The following definition again assumes the Boolean formula B and the corresponding graph of conflicts G_B (again the graph of trivial conflicts G_B^1 can be used).

Definition 2 (Singleton arc-consistency in a SAT instance w.r.t. the graph of conflicts). A literal x_l^k ($l \in \{1,2,\ldots,m_k\}$) from a clause C_k for $k \in \{1,2,\ldots,n\}$ of the formula B is *singleton arc-consistent* with respect to the given graph of conflicts G_B if the formula obtained from B by replacing the clause C_k by the literal x_l^k (the

resulting formula is $(\wedge_{i=1}^{k-1} \vee_{j=1}^{m_i} x_j^i) \wedge x_l^k \wedge (\wedge_{i=k+1}^{n} \vee_{j=1}^{m_i} x_j^i))$ is arc-consistent with respect to the graph of conflicts G_B. □

Unsupported literals in the formula modified by replacing the clause C_k by the literal x_l^k are in conflict with the literal x_l^k. This is quite intuitive, the selection of the literal x_l^k to be assigned the value *true* rules out some other literals. Hence these literals are in conflict with the selected literal x_l^k. Having singleton arc-consistency we are ready to infer new edges for the graph of conflicts.

The intermediate graph of conflicts G_B^2 is constructed from the graph of trivial conflicts G_B^1 in the following way. Initially the graph G_B^2 is identical to the graph G_B^1, that is we start with the initialization $V_B^2 \leftarrow V_B^1$ and $E_B^2 \leftarrow E_B^1$. Then for every literal $y \in V_B^2$ singleton arc-consistency with respect to the graph of conflicts G_B^1 is enforced. If the consistency discovers some unsupported literals, say literals $z_1, z_2, \ldots, z_m$, edges $\{y, z_i\}$ for all $i = 1,2,\ldots,m$ are added into the set of edges E_B^2.

An example of the resulting graph of conflicts is shown in the right part of the figure 1. It is constructed from the original graph of trivial conflicts from the left part of the figure 1. The required complete sub-graphs of the graph are clearly visible.

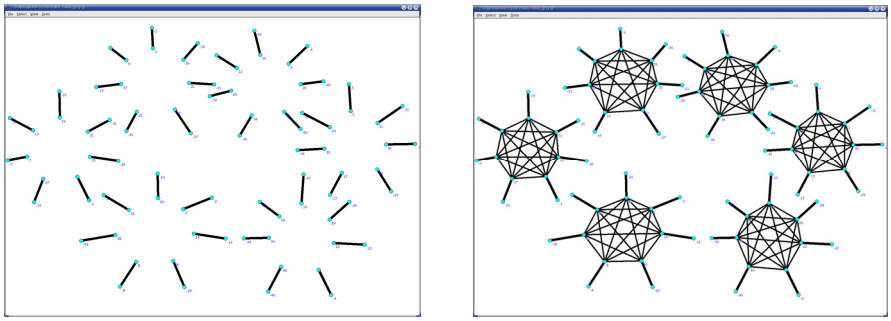

Fig. 1. The left part of the figure shows a graph of trivial conflicts for the SAT benchmark problem pigeon-hole principle number 6 (hole06.cnf). Vertices represents literals, edges are between pairs of positive and negative literals of the same variable. The right part of the figure shows an intermediate graph of conflicts inferred from the original graph of the left by singleton arc-consistency. The graph contains edges from the original graph plus the inferred edges. Six complete sub-graphs each containing seven vertices are clearly visible and can be found by a simple greedy algorithm.

The described process of inference of conflicting literals is relatively generic. Both different initial graphs of trivial conflicts as well as different consistency techniques than arc-consistency and singleton arc-consistency for inference of new edges can be used. Both entities, graphs and consistency techniques, may be considered as parameters of the method.

2.2 Greedy Clique Decomposition and Literal Contribution Counting

To deduce yet more information from the graph of conflicts $G_B^2 = (V_B^2, E_B^2)$ a clique decomposition of the graph is constructed. Formally, a partition of vertices

$V_B^2 = K_1 \cup K_2 \cup \ldots \cup K_s$ such that each set of vertices K_i for $i = 1,2,\ldots,s$ induces a clique over the set of edges E_B^2 and $K_i \cap K_j = \emptyset$ for all $i, j = 1,2,\ldots,s \,\&\, i \neq j$. Let E_{K_i} denotes the set of edges induced by the clique K_i, let E_R denotes the set of edges outside the clique decomposition, that is $E_R = E_B^2 - \bigcup_{i=1}^{s} E_{K_i}$. Our inference method based on literal contribution counting performs best if cliques of the decomposition are as large as possible (that is s must be as small as possible) and the size of E_R is as small as possible. The better the quality of the decomposition is the stronger results are produced by our inference method. Since the problem of finding the optimal clique decomposition with respect to the above criterion is obviously *NP*-complete on a general graph [16], we cannot afford to construct the optimal decomposition and we must abandon this requirement. Nevertheless experiments showed that the simple greedy algorithm can find a clique decomposition of acceptable quality (with respect to clique sizes and the number of edges outside the decomposition).

Our greedy algorithm for finding a clique decomposition is based on the standard greedy algorithm for finding the largest clique. The main loop of the greedy algorithm repeatedly finds a largest clique. The largest clique is found in the following way. A vertex of the highest degree is found in the graph and it is added to the constructed clique which is empty at the beginning. Then the graph is restricted on the neighborhood of the selected vertex and a vertex of the highest degree in this neighborhood is selected as second. Then the graph is again restricted on the neighborhood of these two vertices (that is the considered vertices are neighbors of both the first and the second selected vertex) and the algorithm continues until the neighborhood of selected vertices is empty. The constructed clique and its neighborhood are removed from the graph and the next clique is constructed. This main loop continues until the graph is empty.

The above described greedy algorithm performed over the graph from the right part of the figure 1 finds the clique decomposition consisting of six cliques of size seven. The fact that at most one literal from a clique can be selected to be assigned the value *true* is used in our inference method.

For the following definitions we assume a Boolean formula $B = \wedge_{i=1}^{n} \vee_{j=1}^{m_i} x_j^i$ and the corresponding clique decomposition $V_B^2 = K_1 \cup K_2 \cup \ldots \cup K_s$ of the intermediate graph of conflicts $G_B^2 = (V_B^2, E_B^2)$. Next let $I \subseteq \{1,2,\ldots,n\}$ be a set of indexes of some clauses of the formula B. The set I defines a sub-formula B_I of the formula B, where $B_I = \wedge_{i \in I} C_i$.

Definition 3 (Literal contribution). A *contribution of a literal* y to the sub-formula B_I is defined as the number of clauses of B_I in which the literal y occurs and it is denoted as $c(y, I)$. □

Definition 4 (Clique contribution). A *contribution of a clique* $K \in \{K_1, K_2, \ldots, K_s\}$ to the sub-formula B_I is defined as $\max_{y \in K}(c(y, I))$ and it is denoted as $c(K, I)$. □

The concept of clique contribution is helpful when we are trying to decide whether it is possible to satisfy the sub-formula B_I using the literals from the clique decomposition. If for instance $\sum_{i \in I} c(K_i, I) < |I|$ holds then the sub-formula B_I cannot be satisfied and hence also B cannot be satisfied. Moreover we can handle a more general case as it is described in the following definitions.

Definition 5 (Clique-consistent literal). A literal $y \in K_i$ for $i \in \{1,2,\ldots,n\}$ is said to be *clique-consistent with respect to the sub-formula* B_I if $\sum_{j \in I \,\&\, j \neq i} c(K_j, I) \geq |I| - c(y, I)$. □

Definition 6 (Clique-consistent formula). A formula B is *clique-consistent with respect to the sub-formula* B_I if all the literals of the formula B are clique-consistent with respect to B_I. □

It is easy to see that a clique-inconsistent literal with respect to some sub-formula of B cannot be selected to be assigned the value *true*. Thus such literals can be ruled out from further reasoning. The proof of this claim is provided in the technical report [28]. In addition, this type of consistency is strictly stronger than the discussed unit propagation, arc-consistency and singleton arc-consistency. The proof of this claim is again given in [28].

The remaining question is how to select the described sub-formulas B_I of B which are used for computation of the clique-inconsistent literals. This selection is crucial for the strength of the proposed clique-consistency. It is clear that we need to rule out as many as possible inconsistent literals. As it is impossible to compute the defined consistency with respect to all such sub-formulas of B, because there are too many sub-formulas, we need to select a subset of them carefully. The experiments carried out in [28] showed that a good strength of the clique-consistency can be obtained by selecting clauses into the sub-formula B_I which have the same number of literals. More precisely, we use sub-formulas $B_{I_r} = \wedge_{i \in I_r} C_i$ of B, where $I_r = \{i \in \{1,2,\ldots,n\} \mid m_i = r\}$ for all possible $r \in \mathbb{N}$ for which B_{I_r} is not empty (we suppose that a clause of B does not contain an individual literal more than once). Let us note that we do not know whether this selection is the best possible.

Theorem 1 (Complexity of clique-consistency enforcing algorithm). *There exists a polynomial time algorithm for enforcing clique-consistency with respect to a sub-formula of a given input formula.* ■

The proof of this theorem can be found in [28]. Having such an algorithm it is possible to extend it for multiple sub-formulas B_{I_r} simply by running the algorithm for each $r \in \mathbb{N}$ for which B_{I_r} is non-empty. Since r is proportional to the size of the input the, resulting algorithm is also polynomial.

2.3 Output of the Reformulation Process

At this point everything is ready to introduce the final step of our reformulation method. We will be constructing a modified formula β which is initially set to be identical to B. We will further preprocess B by the singleton version of the defined clique-consistency. Conflicts inferred by this further preprocessing will be stored in a new graph of conflicts $G_B^3 = (V_B^3, E_B^3)$ which is initially set to be the same as the graph G_B^2. The graph G_B^3 will be called a *final graph of conflicts* in this context.

Singleton clique-consistency is computed in the following way. For each literal y of the input formula B we enforce clique-consistency for the formula obtained from B by selecting a literal y to be assigned the value *true*. More precisely, clauses containing y are removed and the negation of the literal y is removed from remaining clauses of B (removal of a literal x_k^i from the clause $C_i = \vee_{j=1}^{m_i} x_j^i$ of the formula

B is defined as replacement of the clause C_i by the clause $(\vee_{j=1}^{k-1} x_j^i) \vee (\vee_{j=k+1}^{m_i} x_j^i))$. The clique-consistency is then enforced for the resulting formula. Some literals may be found inconsistent during consistency enforcing. These literals are in conflict with the literal y. If for some clause all its literals are found inconsistent with y then the literal y cannot be selected to be *true* and a new clause $\neg y$ is added to β ($\beta \leftarrow \beta \wedge \neg y$). Otherwise the conflicting literals are stored in the graph of conflicts G_B^3 as new edges (that is, if the literal y is in conflict with the literal z, the edge $\{y,z\}$ is added to G_B^3).

If for some clause it is discovered by the clique-consistency that none of its literals can be assigned the value *true* the process terminates with the answer that the formula B cannot be satisfied. This outcome is ensured by the correctness of the method. Our experiments showed that this situation is the most successful case, because an answer to the satisfiability is obtained in polynomial time without further expensive search for a solution.

If the process does not terminate with the negative answer then all the edges of the graph of conflicts G_B^3 are translated into new clauses of the formula β. That is, for every edge $\{y,z\} \in E_B^3$ we add a clause $y \vee \neg z$ into the formula β ($\beta \leftarrow \beta \wedge (y \vee \neg z)$). The resulting formula β is equivalent with the original input formula B. Notice that the conflicts inferred by the preceding reformulation stages are also reflected in the formula β, since the graph G_B^3 subsumes the preceding graphs of conflicts G_B^1 and G_B^2. The formula β is finally sent to the SAT solver of the user's choice. Justification of this step is provided by the following corollary of the correctness of the clique-consistency.

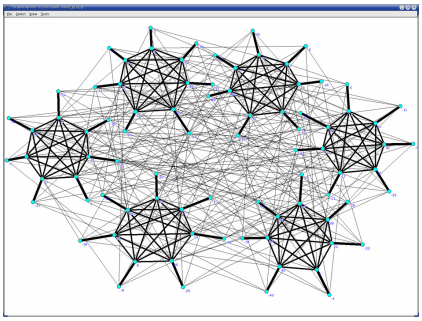

Fig. 2. A final graph of conflicts for the SAT benchmark problem pigeon-hole principle number 6 (hole06.cnf). The graph contains edges from the intermediate graph of conflicts from figure 1 plus the edges inferred by singleton clique-consistency.

Corollary 1 (Correctness of reformulation). *The formula β resulting from the described preprocessing has the same set of satisfying valuations as the original formula B.* ■

The graph of conflicts G_B^3 resulting from processing the intermediate graph of conflicts G_B^2 for the SAT benchmark problem from figure 1 is shown in figure 2.

3 Experimental Results

We chose three state-of-the-art SAT solvers for comparison with our reformulation method. The SAT solvers of our choice were zChaff [14, 21], HaifaSAT [15, 27] and MiniSAT (a version with SATElite preprocessing integrated) [11, 12] (we used the latest available versions to the time of writing this paper). Our choice was guided by the results of several last SAT competitions [18, 25] in which these solvers belonged to the winners. The secondary guidance was that complete source code (in C/C++) for all these solvers is available on web pages of their authors. As we implemented our method in C++ too, this fact allowed us to compile all source codes by the same compiler with the same optimization options which guarantees more equitable conditions for the comparison (a complete source code implementing our method in C++ available at the web page: http://ktiml.mff.cuni.cz/~surynek/software/ssat/ssat.html). All the tests were run on the machine with two AMD Opteron 242 processors (1600 MHz) with 1GB of memory under Mandriva Linux 10.2. Our method as well as the listed SAT solvers were compiled by the gcc compiler version 3.4.3 with options provided maximum optimization for the target testing machine (-O3 -mtune=opteron). Although the testing machine has two processors no parallel processing was used.

3.1 Difficult SAT Instances Selected for Experiments

The testing set consisted of several difficult unsatisfiable SAT instances. This set of benchmark problems was collected by Aloul [1] and it is provided at his research web page. The details about hardness and construction of these instances are discussed in [2], but let us briefly introduce the problems.

Pigeon Hole Instances. *[hole]* This is a standard SAT benchmark encoding the pigeon hole principle problem. The problem asks whether it is possible to place $n+1$ pigeons in n holes without two pigeons being in the same hole. The problem is obviously unsatisfiable. We used six instances of this problem ranging from 6 to 12 holes.

Randomized Urquhart Instances. *[urq]* This set of benchmark problems contains several artificially constructed hard unsatisfiable instances. More details about these problems are provided in [29]. In addition, the problems were randomized for our testing purposes. We used four instances of the problems of this type.

Field Programmable Gate Array Routing Instances. *[fpga, chnl]* This benchmark problem resembles the pigeon hole problem. The question is whether it is possible to route n connections through m tracks provided by the field programmable gate array component. If $n > m$ the problem cannot be satisfied. We used sixteen unsatisfiable instances of this problem for various number of required routes and connections. Two different encodings of the problem are used - denoted *fpga* and *chnl*. More details about the encoding of this problem are provided in [23].

For each benchmark SAT instance we measured the overall time necessary to decide its satisfiability. The results are shown in table 1 and table 2. The speedup obtained by using our method compared to a selected SAT solver is also shown.

Table 1. Experimental comparison of three SAT solvers over the selected difficult benchmark SAT instances. We used the timeout of 10.0 minutes (600.00 seconds) for all the tests.

Instance	Satisfiable	Number of variables / number of clauses	MiniSAT (seconds)	zChaff (seconds)	HaifaSAT (second)
chnl10_11	unsat	220/1122	**34.30**	**7.54**	*> 600.00*
chnl10_12	unsat	240/1344	**101.81**	**9.11**	*> 600.00*
chnl10_13	unsat	260/1586	**200.30**	**11.47**	*> 600.00*
chnl11_12	unsat	264/1476	*> 600.00*	**33.49**	*> 600.00*
chnl11_13	unsat	286/1472	*> 600.00*	**187.08**	*> 600.00*
chnl11_20	unsat	440/4220	*> 600.00*	**329.57**	*> 600.00*
urq3_5	unsat	46/470	**95.04**	*> 600.00*	*> 600.00*
urq4_5	unsat	74/694	*> 600.00*	*> 600.00*	*> 600.00*
urq5_5	unsat	121/1210	*> 600.00*	*> 600.00*	*> 600.00*
urq6_5	unsat	180/1756	*> 600.00*	*> 600.00*	*> 600.00*
hole6	unsat	42/133	**0.01**	**0.01**	**0.01**
hole7	unsat	56/204	**0.09**	**0.04**	**0.02**
hole8	unsat	72/297	**0.49**	**0.23**	**0.94**
hole9	unsat	90/415	**3.64**	**1.46**	**478.16**
hole10	unsat	110/561	**39.24**	**7.53**	*> 600.00*
hole11	unsat	132/738	*> 600.00*	**32.36**	*> 600.00*
hole12	unsat	156/949	*> 600.00*	**372.18**	*> 600.00*
fpga10_11	unsat	220/1122	**44.77**	**12.58**	*> 600.00*
fpga10_12	unsat	240/1344	**119.26**	**33.82**	*> 600.00*
fpga10_13	unsat	260/1586	**362.24**	**76.15**	*> 600.00*
fpga10_15	unsat	300/2130	*> 600.00*	**274.84**	*> 600.00*
fpga10_20	unsat	400/3840	*> 600.00*	**546.00**	*> 600.00*
fpga11_12	unsat	264/1476	*> 600.00*	**55.70**	*> 600.00*
fpga11_13	unsat	286/1742	*> 600.00*	**237.54**	*> 600.00*
fpga11_14	unsat	308/2030	*> 600.00*	*> 600.00*	*> 600.00*
fpga11_15	unsat	330/2340	*> 600.00*	*> 600.00*	*> 600.00*
fpga11_20	unsat	440/4220	*> 600.00*	*> 600.00*	*> 600.00*

Table 2. Experimental comparison of three SAT solvers with the method using clique-consistency over the selected difficult benchmark SAT instances. Again timeout of 10.0 minutes (600.00 seconds) for all the tests was used.

Instance	Decided by preprocessing	Cliques (count x size)	Decision (seconds)	Speedup ratio w.r.t. MiniSAT	Speedup ratio w.r.t zChaff	Speedup ratio w.r.t HaifaSAT
chnl10_11	yes	20 x 11	**0.43**	**79.76**	**17.53**	*> 1395.34*
chnl10_12	yes	20 x 12	**0.60**	**169.68**	**8.51**	*> 1000.00*
chnl10_13	yes	20 x 13	**0.78**	**256.79**	**14.70**	*> 769.23*
chnl11_12	yes	22 x 12	**0.70**	*> 857.14*	**47.84**	*> 857.14*
chnl11_13	yes	22 x 13	**0.92**	*> 652.17*	**203.34**	*> 652.17*
chnl11_20	yes	22 x 20	**5.74**	*> 104.42*	**57.41**	*> 104.42*
urq3_5	no	47 x 2	**130.15**	**0.73**	*N/A*	*N/A*
urq4_5	no	73 x 2	*> 600.00*	*N/A*	*N/A*	*N/A*
urq5_5	no	120 x 2	*> 600.00*	*N/A*	*N/A*	*N/A*
urq6_5	no	179 x 2	*> 600.00*	*N/A*	*N/A*	*N/A*
hole6	yes	6 x 7	**0.01**	**1.0**	**1.0**	**1.0**
hole7	yes	7 x 8	**0.02**	**4.5**	**2.0**	**1.0**
hole8	yes	8 x 9	**0.04**	**12.25**	**5.75**	**23.5**
hole9	yes	9 x 10	**0.08**	**45.5**	**18.25**	**5977.00**
hole10	yes	10 x 11	**0.13**	**301.84**	**57.92**	*> 4615.38*
hole11	yes	11 x 12	**0.20**	*> 3000.00*	**161.8**	*> 3000.00*
hole12	yes	12 x 13	**0.30**	*> 2000.00*	**1240.6**	*> 2000.00*
fpga10_11	yes	20 x 11	**0.46**	**97.32**	**27.34**	*> 1304.34*
fpga10_12	yes	20 x 12	**0.64**	**186.34**	**52.84**	*> 937.50*
fpga10_13	yes	20 x 13	**0.84**	**431.23**	**90.65**	*> 714.28*
fpga10_15	yes	20 x 15	**1.39**	*> 431.65*	**197.72**	*> 431.65*
fpga10_20	yes	20 x 20	**4.72**	*> 127.11*	**115.67**	*> 127.11*
fpga11_12	yes	22 x 12	**0.76**	*> 789.47*	**73.28**	*> 789.47*
fpga11_13	yes	22 x 13	**1.01**	*> 594.05*	**235.18**	*> 594.05*
fpga11_14	yes	22 x 14	**1.30**	*> 461.53*	*> 461.53*	*> 461.53*
fpga11_15	yes	22 x 15	**1.67**	*> 359.28*	*> 359.28*	*> 359.28*
fpga11_20	yes	22 x 20	**5.96**	*> 100.67*	*> 100.67*	*> 100.67*

3.2 Effect of Problem Reformulation

As it is evident from our experiments the proposed method brings significant improvement in terms of time necessary for the decision of the selected difficult benchmark problems (Pigeon hole, FPGA routing instances). The improvements are in the order of magnitude in comparison to all tested state-of-the-art SAT solvers. It seems that the improvement on selected benchmarks is exponential with respect to the best tested SAT solver. The conclusion is that there is still a space to improve SAT solvers. However, the domain of the improvement is more likely in the difficult instances of SAT problems which are typically unsatisfiable. It is also evident that the clique-consistency is not an universal method for difficult SAT instances. There is no improvement on instances where no cliques of reasonable size are found (randomized Urquhart instances). The interesting feature of the tested SAT instances is that they contain cliques of the same size. This may be accounted to the symmetrical formulation of the problems.

In our further experiments we also performed the comparison with the RSAT solver [24]. The results were very similar in the sense that the solver does not cope well with these problems. Unfortunately the solver is provided without the source code so we do not consider this test as a relevant one. Another SAT solver which worth consideration for our tests (achieved good results in the SAT Race competition [25]) - Eureka [22] - is not provided at all (no source code nor executables are provided).

We also tested our approach on SAT instances where the preprocessing stage does not terminate by the answer that the given SAT instance cannot be satisfied. This is the situation when the problem is not decided by the preprocessing stage and a new equivalent SAT instance is produced and sent to the solver. In such situations our method does not provide competitive results. The resulting formula is typically solved faster by the SAT solver but the preprocessing stage takes too much time. The unaffordable time consumption in the preprocessing stage is caused by extensive propagation performed by the method by which huge numbers of conflicts are inferred. It seems that on these problems the proposed approach is too strong and represents an overhead only. The numbers of inferred conflicts is not proportional to the time saved in the search for the solution stage. Moreover, as it was mentioned in the introduction, there is almost no room for improving the SAT solvers on such easy (satisfiable) SAT instances. However, this disadvantage may be overcome firstly by a better implementation of our technique (our current implementation is an experimental prototype and the quality of our code is uncompetitive with the quality of code of the tested SAT solvers) and secondly by making the propagation less extensive on problems with many conflicts (that is, not to infer all the conflicts).

The question may now be what to do with the method at current stage of implementation when we have a new problem of unknown difficulty. That is shall we use the method or the SAT solver of our choice directly? Technically we can answer this question as follows. We can run both the preprocessing method and the SAT solver in parallel. On a machine with more than one processor we obtain an exponential speedup (the method succeeds) or no improvement. On a machine with only one processor we may obtain an exponential speedup at the expense of constant slow-

down. However, the ultimate goal of our implementing efforts is to answer this question automatically within the preprocessing phase.

4 Related Works

Our method for SAT problem reformulation was originally proposed for solving planning problems using planning graphs. It was named projection consistency and it was described in the technical report [28] by Surynek. Clique-consistency proposed in this paper is an adaptation of projection consistency for the SAT domain. In addition to the description of projection consistency, the technical report contains theoretical comparison of the proposed consistency with arc-consistency and singleton arc-consistency (briefly said AC and SAC can be simulated by projection consistency; moreover there are cases on which projection consistency propagates while AC and SAC do not; the similar results hold for clique-consistency too).

The idea of exploiting structural information for solving problems is not new. There is a lot of works concerning this topic. Many of these works are dealing with methods for breaking symmetries [2, 4, 8]. We share the goal with these methods, which is to reduce the search space. However, we differ in the way how we are doing this. We are rather trying to infer what would happen if the search over the problem proceeds in some way. And if that direction seems to be unpromising the corresponding part of the search space is skipped. Symmetry breaking methods are rather trying not to do the same work twice (or more times) by a clever transformation of the original problem.

Our work was much influenced by the paper of Aloul, Markov and Sakallah [2]. We are studying the same set of difficult SAT problems. Nevertheless, it seems that our method is simpler to implement and more effective on the set of selected testing problems.

Another original approach to solving SAT problems is to exploit integer programming (IP) techniques. An interesting combination of IP and SAT techniques is given in [19]. The proposed IP approach is especially successful on difficult SAT problems.

Finally let us note that the detection of cliques in the structure of the problem is not new. A work dealing with a consistency based on cliques of inequalities was published by Sqalli and Freuder [26]. They use information about cliques to reach more global reasoning about the problem. Another work dealing with the similar ideas is [13] in which the authors use a graph structure of the problem to transform it into another formulation based on global constraints, which provide stronger propagation than the original formulation.

5 Conclusions and Future Work

We proposed a method for preprocessing difficult (unsatisfiable) SAT instances based on the greedy clique decomposition of the transformed input CNF formula. Although the method is not universal it provides improvements in the order of magnitude compared to the state-of-the-art SAT solvers on tested SAT instances. Moreover, our

method can be easily integrated into a SAT solver (new or existing) which may significantly improve its performance on difficult SAT instances.

For future we plan to further tune the method to be able to cope better with the problems having few edges in the graphs of conflicts (for example Urquhart instances). This may be done by some alternative consistency technique instead of singleton arc-consistency. We also plan to investigate the possibility to make the preprocessing iterative. That is to further preprocess the formula resulting from the previous preprocessing.

Another issue worth a deeper study is how the cliques of the clique decomposition should look like in order to our method can succeed. Our further experiments showed that better results can be obtained by using a clique decomposition where sizes of the individual cliques differ little (having several cliques of the similar size is better than having one large clique and several much smaller cliques).

We also plan to write an experimental SAT solver which would utilize the clique-consistency during search. This may be useful for early determining that a certain part of the search space does not contain a solution.

Finally an interesting research direction is some kind of a combination of existing symmetry breaking methods and the proposed clique-consistency.

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